Non-amenable C^*-superrigid groups that are not W^*-superrigid

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Group operator algebras

Throughout, G will denote a countable discrete group.

- Left regular representation: $u : G \to \mathcal{U}(\ell^2(G))$
  Where $(u_g\xi)(h) = \xi(g^{-1}h)$ for any $g, h \in G$ and $\xi \in \ell^2(G)$.
- Produces a copy of the group algebra $\mathbb{C}[G] \subset \mathcal{B}(\ell^2(G))$.

The **reduced group C^*-algebra** is given by $C_r^*(G) = \overline{\mathbb{C}[G]}^{\|\cdot\|} \subset \mathcal{B}(\ell^2(G))$.

The **group von Neumann algebra** is $\mathcal{L}(G) = \overline{\mathbb{C}[G]}^{sot} \subset \mathcal{B}(\ell^2(G))$.

$$\mathbb{C}[G] \subset C_r^*(G) \subset \mathcal{L}(G) \subset \mathcal{B}(\ell^2(G))$$

Both $C_r^*(G)$ and $\mathcal{L}(G)$ admit a canonical trace τ extending $\tau(u_g) = \delta_{e,g}$.

Superrigidity for group operator algebras

Theorem (Murray, von Neumann '43)

*$\mathcal{L}(G)$ is a II_1 **factor** if and only if G is **ICC** (infinite conjugacy classes) if and only if there is a unique tracial state on $\mathcal{L}(G)$.*

Question: Can we recover all the information about G from one of $\mathbb{C}[G]$, $C_r^*(G)$ or $\mathcal{L}(G)$?

Definition

A group G is called:

- **ring-superrigid**, if for every group H admitting any $*$-isomorphism $\mathbb{C}[G] \cong \mathbb{C}[H]$ one must have $G \cong H$.
- **C^*-superrigid**, if for every group H admitting any $*$-isomorphism $C_r^*[G] \cong C_r^*[H]$ one must have $G \cong H$.
- **W^*-superrigid**, if for every group H admitting any $*$-isomorphism $\mathcal{L}(G) \cong \mathcal{L}(H)$ one must have $G \cong H$.

Some examples of C^*-superrigid groups

- Any torsion-free abelian group (Scheinberg '74).
- Certain torsion-free virtually abelian groups (Curda, Knuby, Raum, Thiel, White).
- Any torsion-free finitely generated two-step nilpotent group (Eckhardt, Raum '18).
- Any free nilpotent group (Omland '18).
- All 2D crystallographic groups (Chan, Lippert, Moutzouris, Weld '26)

All the examples above are amenable, but the following are famously equivalent:

1. $\mathcal{L}(G) \cong \mathcal{R}$ (the unique hyperfinite II_1 factor)
2. G is ICC and amenable.

Some examples of W^*-superrigid groups

- First examples of W^*-superrigid groups: some wreath product groups (Ioana, Popa, Vaes '10).

- Various other ICC groups constructed through coinduced groups, amalgamated free products, tree groups, and semidirect products with nonamenable cores (Berbec, Vaes ' 13; Berbec '14; Chifan, Ioana '17; Chifan, Diaz-Arias, Drimbe '20 and '21).

- First examples with property (T): certain wreath-like product groups $G \in \mathcal{WR}(A, B \curvearrowright I)$ (Chifan, Ioana, Osin, Sun '21).

In some cases the W^*-superrigidity results can be leveraged to obtain C^*-superrigidity.
This can be done when the group has the unique trace property, or (equivalently) has trivial amenable radical (Breuillard, Kalantar, Kennedy, Ozawa '14).

C^*-superrigid, but not W^*-superrigid?

Closing $\mathbb{C}[G]$ in a weaker topology brings in more elements to the group operator algebra and, heuristically, we should expect it to be harder to tell if $G \cong H$ only by testing if $\mathcal{L}(G) \cong \mathcal{L}(H)$, as opposed to testing if $C_r^*(G) \cong C_r^*(H)$.

- All countably infinite abelian groups G have $\mathcal{L}(G) \cong L^\infty([0,1], \lambda)$. In opposition to the C^*-superrigidity of torsion-free abelian.

All previously known C^*-superrigid groups were either amenable or W^*-superrigid.

But, we should expect there to be many examples of non-amenable groups that are C^*-superrigid but not W^*-superrigid.

Theorem (A. M., Chifan, Fernández Quero '26)

There exists a family of countable groups $\{G_i\}_{i\in I}$ with $|I| = 2^{\aleph_0}$ such that $G_i \ncong G_j$ whenever $i \neq j$, and each group G_i is C^*-superrigid but not W^*-superrigid.

Wreath-like product groups and the class $\mathcal{AT}$

Wreath-like product groups: Take groups A, B and an action $B \curvearrowright I$ on a set. Then $G \in \mathcal{WR}(A, B \curvearrowright I)$, if there is a s.e.s.

$$1 \to A^{(I)} := \bigoplus_I A_i \to G \to B \to 1$$

where $A_i \cong A$ and such that $gA_ig^{-1} = A_{\varepsilon(g)\cdot i}$ for any $i \in I$, $g \in G$.

Definition (The class $\mathcal{AT}$)

Denote by $\mathcal{AT}$ the class of all amalgamated free product groups $G = G_1 *_C G_2$ satisfying the following conditions:

(i) For $j = 1, 2$, $G_j \in \mathcal{WR}(A_j, B_j \curvearrowright I_j)$ has property (T).

(ii) Each A_j is nontrivial abelian, and each B_j is a nontrivial ICC subgroup of a hyperbolic group with $C_{B_j}(b)$ amenable for any $b \in B_j \setminus \{1\}$.

(iii) $B_j \curvearrowright I_j$ satisfies $\mathrm{Stab}_{B_j}(i)$ amenable for every $i \in I_j$.

(v) The common subgroup $C < G_1, G_2$ is icc, non-amenable, with Haagerup property, and it is almost malnormal in both G_1 and G_2.

Product rigidity and W^*-superrigidity in $\mathcal{AT}$

Theorem (A. M., Chifan, Fernández Quero '26)

Let $n \in \mathbb{N}_{\geq 0}$.

For every $1 \leq i \leq n$, let $G_i \in \mathcal{AT}$, set $G = G_1 \times \cdots \times G_n$. Let H be an arbitrary group and assume that $\theta\colon \mathrm{L}(G)^t \to \mathrm{L}(H)$ is a $*$-isomorphism, for some $0 < t \leq 1$. Then $t = 1$ and $G \cong H$. In addition, there exist a group isomorphism $\delta\colon G \to H$, a character $\eta\colon G \to \mathbb{T}$, and a unitary $w \in \mathrm{L}(H)$ such that

$$\theta(u_g) = \eta(g) w v_{\delta(g)} w^*, \text{ for all } g \in G.$$

Lemma

Torsion free groups in $\mathcal{AT}$, and any at most countable sum of these groups satisfy having trivial amenable radical .

Finite products of torsion-free groups in $\mathcal{AT}$ and the products of groups in $\mathcal{C}^0_{AFP}$ (Curda, Drimbe '26) constitute the first examples of C^*-superrigid non-amenable product groups.

Examples of non-amenable C^*-(not W^*-)superrigid groups

For each $n \in \mathbb{N}$, let $G_n \in \mathcal{AT}$ and define $G = \bigoplus_{\mathbb{N}} G_n$.

G is not W^*-superrigid since it is a W^*-McDuff group, i.e. $\mathcal{L}(G)$ is a McDuff II_1 factor.

For any ICC amenable group A we have $\mathcal{L}(G) \cong \mathcal{L}(G) \otimes \mathcal{R} \cong \mathcal{L}(G \times A)$.

Theorem (A. M., Chifan, Fernández Quero '26)

Let H be an arbitrary countable group and $\theta\colon \mathrm{C}^*_r(G) \to \mathrm{C}^*_r(H)$ any $*$-isomorphism. Then, $G \cong H$. Moreover, there exist a group isomorphism $\delta\colon G \to H$, a character $\eta\colon G \to \mathbb{T}$, and an automorphism $\psi \in Aut(\mathrm{C}^*_r(H))$ satisfying

$$\theta(u_g) = \psi(\eta(g)v_{\delta(g)}), \text{ for all } g \in G.$$

Thank you!